

The Large Deviation Principle and Steady-state Fluctuation Theorem for the Entropy Production Rate of a Stochastic Process in Magnetic Fields

Yong CHEN

School of Mathematics and Computing Science, Hunan University of Science and Technology,

Xiangtan, Hunan, 411201, P.R.China.

Hao GE

Beijing International Center for Mathematical Research

and Biodynamic Optical Imaging Center, Peking University

Beijing, 100871, P. R. China.

Jie XIONG

Department of Mathematics, Faculty of Science and Technology, University of Macau,

Taipa, Macau. jiexiong@umac.mo

Lihu XU

Department of Mathematics, Faculty of Science and Technology, University of Macau,

Taipa, Macau. lihuxu@umac.mo

Abstract

Fluctuation theorem is one of the major achievements in the field of nonequilibrium statistical mechanics during the past two decades. Steady-state fluctuation theorem of sample entropy production rate in terms of large deviation principle for diffusion processes have not been rigorously proved yet due to technical difficulties. Here we give a proof for the steady-state fluctuation theorem of a diffusion process in magnetic fields, with explicit expressions of the free energy function and rate function. The proof is based on the Karhunen-Loève expansion of complex-valued

Ornstein-Uhlenbeck process.

Keywords: A Langevin equation in magnetic fields, Entropy production rate (EPR), Large deviation principle (LDP), Fluctuation theorems, Nonequilibrium, Gallavotti-Cohen symmetry, Karhunen-Loève Expansion.

1 Introduction

Nonequilibrium phenomena are widely present in physics, chemical reactions, cellular dynamics and biological systems [32]. To mathematically characterize the ‘distance’ between these systems and equilibrium, the entropy production rate (EPR), as one of the fundamental concept in nonequilibrium thermodynamics, was first proposed at the macroscopic level [33]. Concrete expressions of EPRs in various kinds of dynamics have already been discovered, such as in fluid dynamics, diffusion processes, chemical reactions and reaction-diffusion processes [10, 11, 4]. Bridgman clarified the important equation for “increase of entropy in the region within a closed surface” (dS/dt) in terms of the difference between “entropy which has flowed out of the region across the surface” and “entropy generated by irreversible processes within the surface” which is exactly the EPR [4].

Since 1970s, there has been a growing fascination toward nonequilibrium phenomena in stochastic systems at steady state [17, 18]. Equipped by the mathematical tools in the field of stochastic process, especially the perspective of trajectory, the stochastic theory of nonequilibrium steady state, taking the stationary Markovian processes as the fundamental models, has been significantly developed in the past 40 years. A comprehensive, but rather mathematical monograph has already been published [21], and also has been applied to concrete biophysical systems [41, 16].

The concept of time reversibility in Markovian dynamics, in a statistical sense, for the forward stationary dynamics and its time reversal, is regarded as being equivalent to the concept of equilibrium state in thermodynamics. The idea can even trace back to Kolmogorov [27, 28]. Furthermore, the EPR is defined as the time-averaged relative entropy of the distributions for the forward and reversed processes [21]. Following this definition, a mesoscopic definition of the EPR along a single trajectory (called sample EPR) was proposed, as

the Radon-Nikodym derivative of the distributions for the forward and reversed processes [35, 21].

In 1993, a breakthrough occurred in nonequilibrium statistical mechanics, when Evans, Cohen and Morriss [12] found in computer simulations that the sample EPR of a steady flow has a highly nontrivial symmetry, which is called the fluctuation theorem in the mathematical theory put forward by Gallavotti and Cohen [15]. The fluctuation theorem gives a general formula valid in nonequilibrium systems, for the logarithm of the probability ratio of observing trajectories that satisfy or “violate” the second law of thermodynamics [37].

Kurchan [29] later pointed out that the fluctuation theorem also holds for certain diffusion processes, and Lebowitz and Spohn [31] extended Kurchan’s results to quite general Markov processes, formulated by the large deviation principle (LDP). Both of them used the mesoscopic definition of sample EPR. However, their proof is not mathematically rigorous. In 2003, Jiang and Zhang [22] rigorously proved the steady-state fluctuation theorem of the entropy production for Markovian jumping processes.

This paper is to partially fill in the gap that there is no rigorous steady-state fluctuation theorem of the EPR for diffusion processes. Our model (1) is a Langevin equation which governs the motion of a charged test particle in magnetic field undergoing collisions with particles in medium, we refer readers to [3, Sect. 11.3] for more details. This type of model has been intensively studied [3, 5, 13]. We shall prove in this paper LDP as well as the associated steady-state fluctuation theorem of its sample EPR, and give the explicit expressions of the rate function and the free energy function.

It turns out rather difficult to rigorously prove the LDP and the steady-state fluctuation theorem of the sample EPR for diffusion processes. Recently, Kim had attempted, in [26], to prove the steady-state fluctuation theorem of the sample EPR for diffusion processes both in compact spaces as well as in R^n . Unfortunately the proof in [26] was not rigorous and contained several serious gaps.

There are several classical approaches to studying LDP, among which the most famous are probably Gärtner-Ellis’ theorem [6], Varadhan’s inverse theorem [6], Kifer’s criterion [25, 20] and Wu’s criterion [40]. With a close look at

the sample EPR (4) below, we find that Gärtner-Ellis's theorem are probably the most promising way, because one can use Girsanov transform to remove the stochastic integral in (4). However, essential smoothness of the related Cramér function in Gärtner-Ellis' theorem is usually very hard to verify. Fortunately, for our equation (1), we can apply the Karhunen-Loève expansion of complex-valued Ornstein-Uhlenbeck process to get an explicit form for Cramér function.

It seems not easy to generalize our method to general high-dimensional Ornstein-Uhlenbeck processes. The main reasons are that there seem no general Karhunen-Loève type expansions for high dimension Ornstein-Uhlenbeck processes such that the stochastic coefficients are independent, and that Eq. (1) under a Girsanov transform as (6) leads to a non-stationary or even non-ergodic process.

The organization of the paper is as the following. We give notations, the main results and a strategy of the proofs in Section 2, while study the Karhunen-Loève expansion and related differential equations associated to (1) in Section 3. These results are used in Section 4 to prove LDP as well as the steady-state fluctuation theorem of the sample EPR.

2 Setting and main results

2.1 The stochastic process in magnetic fields and its EPR

Suppose that $B(t) = (B_1(t), B_2(t))$ is a 2-dimensional real standard Brownian motion, we consider the following stationary complex Ornstein-Uhlenbeck process $\{Z_t\}$ arising in magnetic fields:

$$dZ_t = -e^{i\theta} Z_t dt + \sqrt{2 \cos \theta} d\zeta_t^B, \quad t \geq 0, \quad (1)$$

where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\zeta_t^B = \frac{B_1(t) + iB_2(t)}{\sqrt{2}}$ is a complex Brownian motion. Eq. (1) has a unique invariant measure μ with the form (see [3, Eq.(11.20)] or [5, Eq.(1.4)]):

$$d\mu(z) = \frac{1}{\pi} \exp \{-(z_1^2 + z_2^2)\} dz_1 dz_2 = \rho(z) dz_1 dz_2 \quad (2)$$

with $z = z_1 + iz_2$. Z_t describes the motion of a test particle in magnetic fields which undergoes the collisions from particles in medium [3, Sect. 11.3].

Since we can identify the metrics of \mathbb{C} and \mathbb{R}^2 , to study Eq. (1), it is equivalent to investigate the following real Ornstein-Uhlenbeck process [2, 19]:

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = - \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{\cos \theta} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}. \quad (3)$$

From [21], the sample EPR of Eq. (3) is

$$\begin{aligned} e_p(t)(\omega) &:= \frac{1}{t} \log \frac{d\mathbf{P}_{[0,t]}}{d\mathbf{P}_{[0,t]}^-}(\omega) \\ &= \frac{1}{t} \left[\frac{2 \sin^2 \theta}{\cos \theta} \int_0^t |X(s)|^2 ds + \frac{2 \sin \theta}{\sqrt{\cos \theta}} \int_0^t [X_2(s), -X_1(s)] dB(s) \right], \end{aligned} \quad (4)$$

it is also the sample EPR of Eq. (1). The EPR is

$$e_p := \mathbb{E}^\mu[e_p(t)] = \frac{2 \sin^2 \theta}{\cos \theta},$$

where \mathbb{E}^μ is the probability expectation induced by the stationary process X_t .

By ergodic theory [21], we have

$$\lim_{t \rightarrow \infty} e_p(t) = e_p \quad a.s.. \quad (5)$$

Naturally one may ask the convergence speed of (5) and its deviation behavior. Hence, LDP and fluctuation theorem of sample EPR have their importance not only in physics but also in mathematics.

The process is reversible and thus $e_p(t) \equiv 0$ for all $t \geq 0$, if and only if $\theta = 0$ [34]. From now on, we only consider the case

$$\theta \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right).$$

2.2 Cramér function and LDP

Let

$$dW(t) = dB(t) - \frac{2\lambda \sin \theta}{\sqrt{\cos \theta}} [X_2(t), -X_1(t)]' dt.$$

By the well-known Cameron-Martin-Girsanov theorem, $W(t)$ is a standard 2 dimensional Brownian motion under the measure \mathbb{P}_W which is uniquely determined by

$$\frac{d\mathbb{P}_W}{d\mathbb{P}_B} \Big|_{\mathcal{F}_t} = \exp \left\{ -\frac{2\lambda^2 \sin^2 \theta}{\cos \theta} \int_0^t |X(s)|^2 ds + \frac{2\lambda \sin \theta}{\sqrt{\cos \theta}} \int_0^t [X_2(s), -X_1(s)] dB(s) \right\}. \quad (6)$$

Under the new measure \mathbb{P}^W , then Eq. (3) is rewritten as

$$\begin{aligned} \begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} &= - \begin{bmatrix} \cos \theta & -(1+2\lambda) \sin \theta \\ (1+2\lambda) \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} dt + \sqrt{\cos \theta} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} \\ &= \begin{bmatrix} -r & d \\ -d & -r \end{bmatrix} \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} dt + \sqrt{\cos \theta} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}. \end{aligned} \quad (7)$$

μ is also the unique invariant measure of $Y(t)$. Since the process $X(t)$ is stationary, so is $Y(t)$.

Define

$$\Lambda_t(\lambda) = \frac{1}{t} \log \mathbb{E}^\mu \exp \{t\lambda e_p(t)\},$$

by (6) and (7) we have

$$\begin{aligned} \Lambda_t(\lambda) &= \frac{1}{t} \log \mathbb{E}^\mu \exp \left\{ \frac{2\lambda \sin^2 \theta}{\cos \theta} \int_0^t |X(s)|^2 ds + \frac{2\lambda \sin \theta}{\sqrt{\cos \theta}} \int_0^t [X_2(s), -X_1(s)] d\vec{B}(s) \right\} \\ &= \frac{1}{t} \log \mathbb{E}^\mu \exp \left\{ \frac{2 \sin^2 \theta}{\cos \theta} \lambda(1+\lambda) \int_0^t |Y(s)|^2 ds \right\}. \end{aligned} \quad (8)$$

For a given λ , if $\lim_{t \rightarrow \infty} \Lambda_t(\lambda)$ exists, we denote

$$\Lambda(\lambda) := \lim_{t \rightarrow \infty} \Lambda_t(\lambda)$$

and

$$\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}.$$

Λ is called Cramér function [6].

Let us next recall the definition of LDP [6]. Let \mathcal{X} be a complete separable metric space, $\mathcal{B}(\mathcal{X})$ the Borel σ -field of \mathcal{X} , and $\{X_t\}_{t \geq 0}$ a family of stochastic processes valued in \mathcal{X} .

Definition 2.1. $\{X_t\}_{t \geq 0}$ satisfies a large deviation principle (LDP) if there exist a family of positive numbers $\{h(t)\}_{t \geq 0}$ which tends to $+\infty$ and a function $I(x)$ which maps \mathcal{X} into $[0, +\infty]$ satisfying the following conditions:

- (i) for each $l < +\infty$, the level set $\{x : I(x) \leq l\}$ is compact in \mathcal{X} ;
- (ii) for each closed subset F of \mathcal{X} ,

$$\limsup_{t \rightarrow \infty} \frac{1}{h(t)} \log \mathbb{P}(X_t \in F) \leq - \inf_{x \in F} I(x);$$

(iii) for each open subset G of \mathcal{X} ,

$$\liminf_{t \rightarrow \infty} \frac{1}{h(t)} \log \mathbb{P}(X_t \in G) \geq - \inf_{x \in G} I(x),$$

here $h(t)$ is called the speed function and $I(x)$ is the rate function.

2.3 The main results and the strategy for their proofs

Our main results are the following two theorems, the first one is about large deviation, while the second is the steady-state fluctuation theorem.

Theorem 2.2 (LDP). *The sample EPR $e_p(t)$ defined by (4), satisfies an LDP with $h(t) = t$ in Definition 2.1. The rate function is*

$$I(x) = \begin{cases} -\frac{1}{2}(1 - \sqrt{\frac{x^2(1+2rc)c}{x^2c+2r}})x + \frac{r}{\pi} \int_0^\infty \log(1 - \frac{x^2c^2-1}{(2r+x^2c)(1+y^2)c})dy, & x \geq 0, \\ -\frac{1}{2}(1 + \sqrt{\frac{x^2(1+2rc)c}{x^2c+2r}})x + \frac{r}{\pi} \int_0^\infty \log(1 - \frac{x^2c^2-1}{(2r+x^2c)(1+y^2)c})dy, & x < 0, \end{cases} \quad (9)$$

where $c = \frac{\cos \theta}{2 \sin^2 \theta}$.

Theorem 2.3 (Fluctuation theorem). *The Cramér's function (or say: free energy function) of $\{e_p(t) : t \geq 0\}$ is*

$$\Lambda(\lambda) = \begin{cases} -F(\ell), & \text{for } \lambda \in \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}\right), \\ \infty, & \text{for } \lambda \notin \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}\right), \end{cases} \quad (10)$$

where

$$\ell = \frac{2 \sin^2 \theta}{\cos \theta} \lambda(1 + \lambda), \quad (11)$$

and for $\ell < \frac{r}{2}$,

$$F(\ell) = \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 y^2})dy. \quad (12)$$

Furthermore, $\Lambda(\lambda)$ and $I(x)$ have the following properties:

$$\Lambda(\lambda) = \Lambda(-(1 + \lambda)), \quad \forall \lambda \in \mathbb{R}^1, \quad I(x) = I(-x) - x, \quad \forall x \in \mathbb{R}. \quad (13)$$

We shall prove the above theorems by finding an explicit form of Cramér function Λ and checking that it satisfies the three conditions in Gärtner-Ellis' Theorem. We shall use the following strategy to figure out the explicit form of Cramér function:

(1). We rewrite Eq. (7) as a complex stochastic equation by setting $Z_t = Y_1(t) + iY_2(t)$.

(2). Under the framework of complex-normal system, by Karhunen-Loève expansion we write $(Z_t)_{0 \leq t \leq T}$ under a special orthonormal basis $\{e_k\}_{k \geq 1}$ of $L^2([0, T]; \mathbb{C})$ as $Z_t = \sum_{k \geq 1} w_k e_k(t)$ such that $\{w_k\}_{k \geq 1}$ are independent normal-distributed sequence. Hence,

$$\mathbb{E} \exp\left\{\int_0^T |Z_s|^2 ds\right\} = \prod_{k \geq 1} \mathbb{E} e^{|w_k|^2} \quad (14)$$

(3). To find the special orthonormal basis $\{e_k\}_{k \geq 1}$ in (2), we need to solve eigenvalues problem of a trace class integral operator

$$Kf(s) = \int_0^T R(t, s)f(s)dt, \quad f \in L^2([0, T]; \mathbb{C})$$

with $R(t, s)$ is a kernel satisfying $R(s, t) = \overline{R(t, s)}$. This problem turns out to look for solutions of a complex ordinary differential equations.

(4). We demonstrate the argument in (3) when $\{Z_t : 0 \leq t \leq T\}$ is a real normal system. To look for the special orthonormal basis $\{e_k\}_{k \geq 0}$ in $L^2([0, T]; \mathbb{R})$ such that $Z_t = \sum_{k \geq 1} w_k e_k(t)$ and w_1, w_2, \dots are independent, we only need to check

$$\mathbb{E}\langle Z, e_i \rangle \langle Z, e_j \rangle = 0 \quad \forall i \neq j,$$

i.e.,

$$\int_0^T \int_0^T \mathbb{E}(Z_s Z_t) e_i(s) ds e_j(t) dt = 0. \quad (15)$$

Let $R(t, s) = \mathbb{E}(Z_s Z_t)$ and $Ke_i(t) = \int_0^T R(t, s)e_i(s)ds$, as long as e_i is an eigenfunction of K with λ_i , i.e.,

$$Ke_i = \lambda_i e_i, \quad (16)$$

(15) immediately follows. Eq. (16) is equivalent to solve an ordinary differential equation [30, Chapter 1].

However, the expansion for a complex-normal system is much more complicated than the above real normal case, we need to use Karhunen-Loève expansion.

3 Karhunen-Loéve expansion of complex-valued Ornstein-Uhlenbeck process

3.1 Some preliminary of Karhunen-Loéve expansion

In this section, we adopt the formulation and notations of [19] and [2, Chapter 1].

Definition 3.1. Let $\xi = \xi_1 + i\xi_2$ be a complex random variable. If ξ_1 and ξ_2 are independent real random variables and subject to the same distribution $N(0, \frac{a}{2})$ with $a \geq 0$, ξ is called a complex-normal random variable and we denote it by $\mathcal{CN}(0, a)$. A system of complex random variables $\Xi = \{\xi_\lambda, \lambda \in \Lambda\}$ is called complex-normal if every linear combination of Ξ is a complex-normal variable.

Remark 1. The above definition is from [19], which is more strict than the complex jointly Gaussian random variables given in [2, p19]. This difference ensure the independent property in the Karhunen-Loéve expansion.

We cite the following well-known facts about the Karhunen-Loéve expansion [2, Theorem 1.4.1] for further use.

Theorem 3.2. Let $\{X_t, a \leq t \leq b\}$, a, b finite, be a complex-valued L^2 process with zero mean and continuous covariance $R(\cdot, \cdot)$. The integral operator K associated with R is

$$Kf(t) = \int_a^b R(t, s)f(s)ds, \quad s \in [a, b], \quad f \in L^2([a, b], \mathbb{C}). \quad (17)$$

Let $\{e_k, k = 1, 2, \dots\}$ be an orthonormal basis for the space spanned by the eigenfunctions of the nonzero eigenvalues of the operator K , with e_k taken as an eigenfunction corresponding to the eigenvalue λ_k . Then

$$X_t = \sum_{k \geq 1} w_k e_k(t), \quad a \leq t \leq b, \quad (18)$$

where $w_k = \int_a^b X_t \bar{e}_k(t)dt$, and the w_k are orthogonal random variables with

$$\mathbb{E}[w_k] = 0, \mathbb{E}[|w_k|^2] = \gamma_k. \quad (19)$$

The series converges in L^2 , uniformly in t . If the original process X_t is a complex-normal system, then the variables w_k are complex-normal and are stochastically independent, which implies that the above series also converges almost surely.

In case of real normal system, the following type corollary is well-known ([9]). Now we give a version for the complex-normal system. Note that $\gamma_k \geq 0$ for all $k \geq 1$ in Theorem 3.2. Without loss of generality, we assume that $\gamma_1 \geq \gamma_2 \geq \dots$

Corollary 3.3. *Let the complex valued process $\{X_t : a \leq t \leq b\}$ be complex normal and let $\{\gamma_k : k \geq 1\}$ be the eigenvalues defined by (19) with order $\gamma_1 \geq \gamma_2 \geq \dots$. Then, for all $z \in \mathbb{C}$, we have*

$$\mathbb{E}[\exp \left\{ z \int_a^b |X(t)|^2 dt \right\}] = \begin{cases} \prod_{k=1}^{\infty} \frac{1}{1 - z\gamma_k}, & \text{for } \Re z < \frac{1}{\gamma_1}, \\ \infty, & \text{for } \Re z \geq \frac{1}{\gamma_1}, \end{cases} \quad (20)$$

where $\Re z$ is the real part of z .

Proof. It is easy to see from (19) that $\gamma_k \geq 0$ for all $k \geq 1$. Since $w_k, k \geq 1$ have distribution $\mathcal{CN}(0, \gamma_k)$ and are stochastically independent, we have

$$\mathbb{E}[\exp \{z D^2\}] = \prod_{k=1}^{\infty} \mathbb{E}[\exp \{z |w_k|^2\}].$$

On the other hand, it is easy to check

$$\mathbb{E}[\exp \{z |w_k|^2\}] = \left\{ \mathbb{E}[\exp \{z \Re(w_k)^2\}] \right\}^2 = \begin{cases} \frac{1}{1 - z\gamma_k}, & \text{for } \Re z < \frac{1}{\gamma_k}, \\ \infty, & \text{for } \Re z \geq \frac{1}{\gamma_k}. \end{cases}$$

Combining the previous two relations immediately yields (27). \square

3.2 The covariance function and the associated integral operator K_T

We rewrite (7) as a complex-valued Ornstein-Uhlenbeck process

$$dZ_t = -\alpha Z_t dt + \sqrt{2 \cos \theta} d\zeta_t, \quad (21)$$

where $\alpha = r + id$ and $\zeta_t = \frac{W_1(t) + iW_2(t)}{\sqrt{2}}$. Note that $Z_0 \sim \mu$ and Z_t is a stationary process. It is easy to see that Y_t and Z_t have the same distribution since we can identify \mathbb{R}^2 with \mathbb{C} [2, 19]. Hence, by (8) we have

$$\Lambda_t(\lambda) = \frac{1}{t} \log \mathbb{E}^\mu \exp \left\{ \frac{2 \sin^2 \theta}{\cos \theta} \lambda (1 + \lambda) \int_0^t |Z(s)|^2 ds \right\}. \quad (22)$$

It is easy to check

$$Z_t = e^{-\alpha t} \left(Z_0 + \sqrt{2 \cos \theta} \int_0^t e^{\alpha s} d\zeta_s \right). \quad (23)$$

Lemma 3.4. $\{Z_t : 0 \leq t \leq T\}$ is a complex-normal system and has a continuous covariance function $R : [0, T] \times [0, T] \rightarrow \mathbb{C}$ with the form:

$$\begin{aligned} R(s, t) &= e^{-\alpha(s-t)}, & 0 \leq t \leq s \leq T; \\ R(s, t) &= e^{\bar{\alpha}(s-t)}, & 0 \leq s \leq t \leq T. \end{aligned} \quad (24)$$

Proof. Let $n \in \mathbb{N}$, take a sequence of points $0, \frac{T}{2^n}, \frac{T}{2^{n-1}}, \dots, \frac{(2^n-1)T}{2^n}, T$ and define

$$Z_t^n = e^{-\alpha t} Z_0 + e^{-\alpha t} \sum_{k=1}^{2^n} \int_{\frac{(k-1)T}{2^n} \wedge t}^{\frac{kT}{2^n} \wedge t} e^{\alpha s} d\zeta_s.$$

It is easy to see that Z_t^n converges to Z_t in probability for every $t \in [0, T]$.

Since Z_0 are independent of ζ_t ,

$$\left\{ Z_0, \int_0^{\frac{T}{2^n}} e^{\alpha s} d\zeta_s, \int_{\frac{T}{2^n}}^{\frac{T}{2^{n-1}}} e^{\alpha s} d\zeta_s, \dots, \int_{\frac{(2^n-1)T}{2^n}}^T e^{\alpha s} d\zeta_s \right\}$$

is a complex-normal system. By [19, Theorem 2.2], we immediately obtain that $\{Z_t : 0 \leq t \leq T\}$ is a complex-normal system.

It is well known that the covariance function associated to Z_t is

$$R(s, t) = \mathbb{E}[Z_s \overline{Z_t}].$$

When $h = s - t \geq 0$, by stationarity of Z_t and the easy fact $\mathbb{E}[Z_t | Z_0] = e^{-\alpha t} Z_0$, we have

$$R(s, t) = \mathbb{E}[Z_s \overline{Z_t}] = \mathbb{E}[Z_h \overline{Z_0}] = \mathbb{E}[\mathbb{E}[Z_h | Z_0] \overline{Z_0}] = e^{-\alpha h} \mathbb{E}[Z_0 \overline{Z_0}] = e^{-\alpha h}.$$

When $h = s - t < 0$, similarly we have

$$R(s, t) = e^{\bar{\alpha}h}.$$

□

Define an operator K_T on $L^2([0, T]; \mathbb{C})$ by (see [2, p38])

$$K_T f(s) = \int_0^T R(s, t) f(t) dt, \quad f \in L^2([0, T]), \quad (25)$$

where

$$R(s, t) = \begin{cases} e^{-\alpha(s-t)}, & s \geq t, \\ e^{\bar{\alpha}(s-t)}, & s < t, \end{cases} \quad (26)$$

Since $R(t, s)$ is continuous on $[0, T] \times [0, T]$, it is well-known that K_T is a positive self-adjoint trace class operator (see [38, Theorem 3.9]), therefore K_T has discrete real eigenvalues (counting the multiple eigenvalues)

$$\gamma_{T,1} \geq \dots \geq \gamma_{T,n} \geq \dots$$

such that

$$\text{Tr}(K_T) = \sum_{i \geq 1} \gamma_{T,i} = \int_0^T R(t, t) dt = T < \infty.$$

Corollary 3.5. *We have*

$$\mathbb{E} \exp \left\{ z \int_0^T |Z_t|^2 dt \right\} = \begin{cases} \prod_{k=1}^{\infty} \frac{1}{1 - z \lambda_k}, & \text{for } \Re z < \frac{1}{\gamma_{T,1}}, \\ \infty, & \text{for } \Re z \geq \frac{1}{\gamma_{T,1}}. \end{cases} \quad (27)$$

Proof. Since $(Z_t)_{0 \leq t \leq T}$ is a complex-normal system, Corollary 3.3 immediately yields the desired result. \square

3.3 An ordinary differential equation associated to the eigenvalues problem of K_T

The previous proposition only shows that K_T is uniformly bounded, to apply Theorem 3.2, we need to find the eigenvalues $\{\gamma_{i,T}\}_{i \geq 1}$ of K_T and get more information about eigenvalues. The following lemma is important for solving Eq. (31) below.

Lemma 3.6. *For every $T \in (0, \infty)$, the operator norm of K_T satisfies that*

$$\|K_T\| \leq \frac{2}{r}. \quad (28)$$

Thus, the eigenvalues of K_T satisfy

$$\gamma_{T,i} \leq \frac{2}{r}, \quad \forall i, \quad \forall T > 0. \quad (29)$$

Proof. Without loss of generality, we extend the functions R and f to be new functions \tilde{R} and \tilde{f} as below:

$$\tilde{f}(t) = f(t), \quad t \in [0, T]; \quad \tilde{f}(t) = 0, \quad t \notin [0, T];$$

$$\tilde{R}(s, t) = R(s, t), \quad (s, t) \in [0, T]^2; \quad \tilde{R}(s, t) = 0, \quad (s, t) \notin [0, T]^2.$$

Then,

$$\begin{aligned}
\int_0^T \left| \int_0^T R(s, t) f(t) dt \right|^2 ds &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \tilde{R}(s, t) \tilde{f}(t) dt \right|^2 ds \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \tilde{R}(s, s-u) \tilde{f}(s-u) du \right|^2 ds \\
&\leq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} e^{-r|u|} |\tilde{f}(s-u)| du \right|^2 ds \\
&\leq \int_{-\infty}^{\infty} \frac{2}{r} \left(\int_{-\infty}^{\infty} e^{-r|u|} |\tilde{f}(s-u)|^2 du \right) ds \\
&= \frac{2}{r} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\tilde{f}(s-u)|^2 ds \right) e^{-r|u|} du \\
&\leq \frac{4}{r^2} \|f\|_{L^2}^2.
\end{aligned} \tag{30}$$

where the second inequality is by Jessen inequality. Hence, (28) is proved. \square

Remark 2. It follows from Proposition 3.8 that for every $T > 0$, $\|K_T\| = \frac{2}{r}$.

Following the spirit in [36, p113], we derive a differential equation from which all the eigenvalues and eigenfunctions of K_T can be found. More precisely, we have

Proposition 3.7. *Let the operator K_T be (25)-(26). Then $\text{Ker}(K_T) = \{0\}$ and if $K_T f = \gamma f$ with $\gamma \neq 0$ then f is a solution on $[0, T]$ of the differential equation*

$$\gamma f'' + 2id\gamma f' + (2r - \gamma|\alpha|^2)f = 0, \tag{31}$$

subject to the boundary conditions

$$\begin{cases} \bar{\alpha}f(0) - f'(0) = 0, \\ \alpha f(T) + f'(T) = 0. \end{cases} \tag{32}$$

Proof. Our proof follows the spirit in [36, p113]. For any $f \in L^2([0, T]; \mathbb{C})$, it follows from (25) that

$$K_T f(s) = e^{-\alpha s} \int_0^s e^{\alpha t} f(t) dt + e^{\bar{\alpha} s} \int_s^T e^{-\bar{\alpha} t} f(t) dt. \tag{33}$$

Thus, $K_T f(s)$, $s \in [0, T]$ is absolutely continuous. By differentiating both sides of (33) with respect to s , we obtain

$$(K_T f)' = -\alpha e^{-\alpha s} \int_0^s e^{\alpha t} f(t) dt + \bar{\alpha} e^{\bar{\alpha} s} \int_s^T e^{-\bar{\alpha} t} f(t) dt. \tag{34}$$

Similarly, $(K_T f)'(s)$, $s \in [0, T]$ is absolutely continuous and we have that

$$(K_T f)'' = \alpha^2 e^{-\alpha s} \int_0^s e^{\alpha t} f(t) dt + \bar{\alpha}^2 e^{\bar{\alpha} s} \int_s^T e^{-\bar{\alpha} t} f(t) dt - (\alpha + \bar{\alpha}) f(s). \quad (35)$$

It follows from (33) and (34) that

$$e^{-\alpha s} \int_0^s e^{\alpha t} f(t) dt = \frac{\bar{\alpha} K_T f - (K_T f)'}{\bar{\alpha} + \alpha}, \quad e^{\bar{\alpha} s} \int_s^T e^{-\bar{\alpha} t} f(t) dt = \frac{\alpha K_T f + (K_T f)'}{\bar{\alpha} + \alpha}. \quad (36)$$

Substituting it into (35), we have that

$$(K_T f)'' = |\alpha|^2 K_T f + (\bar{\alpha} - \alpha)(K_T f)' - (\alpha + \bar{\alpha})f. \quad (37)$$

Since $\alpha + \bar{\alpha} > 0$, $K_T f(t) = 0$ and (37) imply that $f(t) \equiv 0$ in $L^2([0, T]; \mathbb{C})$.

If $K_T f = \gamma f$ with $\gamma \neq 0$ then substituting $s = 0$ and $s = T$ to (36) implies that $\bar{\alpha} f(0) - f'(0) = 0$, $\alpha f(T) + f'(T) = 0$. And (37) implies that (32) hold. \square

Note that $\alpha = r + id$, it follows from Eq. (31) that the general solution of equation (31) is

$$f(s) = c_1 e^{-i(d+\omega)s} + c_2 e^{-i(d-\omega)s}, \quad (38)$$

where c_1, c_2 are constants and $\omega = \sqrt{\frac{2r}{\gamma} - r^2}$. By (28),

$$\omega = \sqrt{\frac{2r}{\gamma} - r^2} \geq 0 \quad (39)$$

The boundary condition (32) gives

$$\begin{cases} (r + i\omega)c_1 + (r - i\omega)c_2 = 0, \\ (r - i\omega)e^{-i\omega T}c_1 + (r + i\omega)e^{i\omega T}c_2 = 0. \end{cases} \quad (40)$$

In order to have constants c_1 and c_2 such that $c_1^2 + c_2^2 \neq 0$, we need

$$\frac{r + i\omega}{(r - i\omega)e^{-i\omega T}} = \frac{r - i\omega}{(r + i\omega)e^{i\omega T}},$$

i.e.

$$e^{2i\omega T} = \left(\frac{r - i\omega}{r + i\omega}\right)^2, \quad (41)$$

where ω is the unknown in Eq. (41). Eqs. (41) and (39) imply that ω satisfies

$$\frac{\omega}{r} = \cot \frac{\omega T}{2}, \quad \omega \geq 0, \quad (42)$$

or

$$\frac{\omega}{r} = -\tan \frac{\omega T}{2}, \quad \omega \geq 0. \quad (43)$$

Note that $\cot \frac{\omega T}{2}$ and $-\tan \frac{\omega T}{2}$ are both periodical on $(-\infty, \infty)$ with the same period $\frac{2\pi}{T}$. Further observe that both $\cot \frac{\omega T}{2}$ and $-\tan \frac{\omega T}{2}$ have a single intersection point with the line $\frac{\omega}{r}$ in each period, and that intersection points are roots of the equations. Denote by ω_j the j th (positive) root of Eq. (42) and by $\tilde{\omega}_j$ the j th (positive) root of the transcendental equation (43). The relations of ω_j are as follows:

$$\tilde{\omega}_0 = 0 < \omega_1 < \frac{\pi}{T} < \tilde{\omega}_1 < \frac{2\pi}{T} < \omega_2 < \frac{3\pi}{T} < \tilde{\omega}_2 < \frac{4\pi}{T} < \dots \quad (44)$$

From (39) and (44), it is clear that all the eigenvalues are single

We conclude these results in the following proposition.

Proposition 3.8. *Let the operator K_T be as in (25)-(26). Then K_T only has point spectrum, and the eigenvalues are*

$$\gamma_{T,j} = \begin{cases} \frac{2r}{r^2 + \tilde{\omega}_{(j-1)/2}^2}, & 2 \nmid j, \\ \frac{2r}{r^2 + \omega_{j/2}^2}, & 2 \mid j, \end{cases} \quad (45)$$

where $\omega_j, j > 0$ and $\tilde{\omega}_j, j \geq 0$ are the positive roots of the two transcendental equations (42) and (43) respectively, such that

$$\tilde{\omega}_0 = 0 < \omega_1 < \frac{\pi}{T} < \tilde{\omega}_1 < \frac{2\pi}{T} < \omega_2 < \frac{3\pi}{T} < \tilde{\omega}_2 < \frac{4\pi}{T} < \dots \quad (46)$$

In particular, $\gamma_{T,1} = \frac{2}{r}$. Furthermore, the associated eigenfunction of $\gamma_{T,j}$ is

$$f_j(s) = \begin{cases} e^{-isd} [r \sin(s\tilde{\omega}_{(j-1)/2}) + \tilde{\omega}_{(j-1)/2} \cos(s\tilde{\omega}_{(j-1)/2})], & 2 \nmid j, \\ e^{-isd} [r \sin(s\omega_{j/2}) + \omega_{j/2} \cos(s\omega_{j/2})], & 2 \mid j. \end{cases} \quad (47)$$

Proof. (45) and (46) are immediate from the analysis above and (39). Routine calculus based upon (33), in combination with (38) and (40), gives (47) \square

Remark 3. The positive zeros of two transcendental equations (42) and (43) are similar to those of the famous Bessel functions $J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}(\frac{\sin z}{z} - \cos z)$ and $J_{-\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}}(\sin z + \frac{\cos z}{z})$.

4 Proofs of the main results

Recall that in Theorem 2.3 we defined

$$F(\ell) = \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}) dy, \quad \ell \in (-\infty, \frac{r}{2}). \quad (48)$$

Let us now show that it is well defined. It follows from the well-known inequality

$$x \leq -\log(1-x) \leq \frac{x}{1-x}, \quad \forall x < 1, \quad (49)$$

that we have

$$\left| \log\left(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}\right) \right| \leq \begin{cases} \frac{2|\ell|r}{r^2 + \pi^2 y^2}, & \ell \leq 0, \\ \frac{2|\ell|r}{r(r-2\ell) + \pi^2 y^2}, & 0 < \ell < \frac{r}{2}, \end{cases}$$

which implies that

$$|F(\ell)| \leq \begin{cases} \int_0^\infty \frac{2|\ell|r}{r^2 + \pi^2 y^2} dy = |\ell|, & \ell \leq 0 \\ \int_0^\infty \frac{2|\ell|r}{r(r-2\ell) + \pi^2 y^2} dy = \sqrt{\frac{r}{r-2\ell}} \ell, & 0 < \ell < \frac{r}{2} \end{cases} < \infty.$$

To apply Gartner-Ellis Theorem, we shall make use of the following lemma.

Lemma 4.1. *The function F defined by Eq. (48) is well defined. Moreover, we have*

$$F'(\ell) = -\sqrt{\frac{r}{r-2\ell}}, \quad (50)$$

with $\lim_{\ell \rightarrow \frac{r}{2}-} |F'(\ell)| = \infty$.

Proof. We only need to prove (50) Write

$$h(y, \ell) := \log\left(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}\right),$$

for any $a \neq 0$, by the inequality (49), we have

$$\begin{aligned} \left| \frac{h(y, \ell+a) - h(y, \ell)}{a} \right| &= \left| \frac{1}{a} \log\left(1 - \frac{2ar}{r(r-2\ell) + \pi^2 y^2}\right) \right| \\ &\leq \begin{cases} \frac{2r}{r(r-2\ell) + \pi^2 y^2}, & a < 0 \\ \frac{2r}{\frac{1}{2}r(r-2\ell) + \pi^2 y^2}, & 0 < a < \frac{r-2\ell}{4}, \end{cases} \end{aligned}$$

where both the right hand functions are integrable on $y \in [0, \infty)$. Therefore, Lebesgue dominated convergence theorem yields

$$F'(\ell) = \int_0^\infty \frac{\partial}{\partial \ell} h(y, \ell) dy = - \int_0^\infty \frac{2r}{r(r-2\ell) + \pi^2 y^2} dy = -\sqrt{\frac{r}{r-2\ell}}.$$

□

Proposition 4.2. *We write*

$$\ell = \frac{2 \sin^2 \theta}{\cos \theta} \lambda (1 + \lambda).$$

As long as $\ell < \frac{r}{2}$, i.e.,

$$\lambda \in \left(-\frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right), \quad (51)$$

we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}^\mu \exp \left(\ell \int_0^T |Z_t|^2 dt \right) = - \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 x^2}) dx < \infty. \quad (52)$$

As $\ell \geq \frac{r}{2}$, i.e.,

$$\lambda \notin \left(-\frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right),$$

we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}^\mu \exp \left(\ell \int_0^T |Z_t|^2 dt \right) = \infty. \quad (53)$$

Proof. Since $\ell < \frac{r}{2}$, it follows from Corollary 3.5 that

$$\frac{1}{T} \log \mathbb{E}^\mu \exp \left(\ell \int_0^T |Z_t|^2 dt \right) = -\frac{1}{T} \sum_{k \geq 1} \log(1 - \ell \gamma_{T,k}). \quad (54)$$

By (45) and (44), we have

$$\sum_{k \geq 1} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2}) \leq \sum_{k \geq 2} \log(1 - \ell \gamma_{T,k}) \leq \sum_{k \geq 0} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2})$$

for $\ell \leq 0$, and

$$\sum_{k \geq 0} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2}) \leq \sum_{k \geq 2} \log(1 - \ell \gamma_{T,k}) \leq \sum_{k \geq 1} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2})$$

for $\ell \in (0, \frac{r}{2})$.

As $T \rightarrow \infty$, by Lemma 4.1 we have

$$\begin{aligned} \frac{1}{T} \sum_{k \geq 1} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2}) &\rightarrow \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}) dy, \\ \frac{1}{T} \sum_{k \geq 0} \log(1 - \ell \frac{2r}{r^2 + \pi^2 (\frac{k}{T})^2}) &\rightarrow \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}) dy. \end{aligned}$$

Therefore, as $T \rightarrow \infty$,

$$-\frac{1}{T} \sum_{k \geq 1} \log(1 - \ell \gamma_{T,k}) \rightarrow - \int_0^\infty \log(1 - \frac{2\ell r}{r^2 + \pi^2 y^2}) dy.$$

Now we show the second limit. Since $\ell \geq \frac{r}{2} = (\gamma_{T,1})^{-1}$, it follows from Corollary 3.5 that for any $T > 0$,

$$\frac{1}{T} \log \mathbb{E}^\mu \exp \left(\ell \int_0^T |Z_t|^2 dt \right) = \infty, \quad (55)$$

which immediately gives the second limit. \square

Proofs of Theorems 2.2 and 2.3. Recall the relation (22). (10) follows from Proposition 4.2. Since $\ell(\lambda) = \frac{2\sin^2 \theta}{\cos \theta} \lambda(1 + \lambda)$, we have that $\ell(\lambda) = \ell(-(1 + \lambda))$ and

$$\begin{aligned} \Lambda(\lambda) &= \begin{cases} -F(\ell(\lambda)), & \text{for } \lambda \in \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right), \\ \infty, & \text{for } \lambda \notin \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right), \end{cases} \\ &= \begin{cases} -F(\ell(-(1 + \lambda))), & \text{for } \lambda \in \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right), \\ \infty, & \text{for } \lambda \notin \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right), \end{cases} \\ &= \Lambda(-(1 + \lambda)), \end{aligned}$$

Since $\theta \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, Proposition 4.2 implies

$$\mathcal{D}_\lambda \triangleq \{\lambda : \Lambda(\lambda) < \infty\} = \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right),$$

From Lemma 4.1, for $\lambda \in \mathcal{D}_\lambda$ we have

$$\Lambda'(\lambda) = \frac{2\sin^2 \theta}{\cos \theta} (1 + 2\lambda) \sqrt{\frac{r}{r - 2\ell}}.$$

As $\lambda \in \partial \mathcal{D}_\lambda$, $\ell \rightarrow \frac{r}{2}-$ and thus $\Lambda'(\lambda) \rightarrow \infty$. Hence, the Cramér function $\Lambda(\cdot)$ is essentially smooth.

By Gärtner-Ellis theorem [6] implies that the LDP holds with the good rate functions $I(x)$ such that

$$\begin{aligned} I(x) &= \sup \left\{ x\lambda + F(\ell(\lambda)) : \lambda \in \left(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}}, -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r \cos \theta}{\sin^2 \theta}} \right) \right\} \\ &= \sup \left\{ x\lambda(\ell) + F(\ell) : -\frac{\sin^2 \theta}{2\cos \theta} \leq \ell < \frac{r}{2} \right\}, \end{aligned}$$

where $F(\ell)$ is given as (48) and $\lambda(\ell)$ is the inverse function of $\ell = \frac{2\sin^2 \theta}{\cos \theta} \lambda(1 + \lambda)$. By differentiating the function $s(\ell) := x\lambda(\ell) + F(\ell)$, we have that the unique zero point is

$$\ell_0 = \frac{(x^2 c^2 - 1)r}{2(x^2 c + 2r)c}$$

with $c = \frac{\cos \theta}{2 \sin^2 \theta}$. Substituting it into $s(\ell)$, we obtain (9) since $\lim_{\ell \rightarrow \frac{\pi}{2}-} F(\ell) = -\infty$.

We have proved above

$$\Lambda(\lambda) = \Lambda(-(1 + \lambda)),$$

which implies that $I(x) = I(-x) - x$ similar to the proof of [22, Theorem 2.4]. We can also derive $I(x) = I(-x) - x$ from (9) by a direct but complicated calculation. \square

Acknowledgements This work was partly supported by NSFC(No.11101137) and Hunan Provincial NSFC(No. 2015JJ2055). H. Ge is supported by NSFC(No.21373021), and the Foundation for Excellent Ph.D. Dissertation from the Ministry of Education in China (No. 201119). J. Xiong was supported by Macao Science and Technology Fund FDCT 076/2012/A3 and Multi-Year Research Grants of the University of Macau Nos. MYRG2014-00015-FST and MYRG2014-00034-FST. L. Xu is supported by the grant SRG2013-00064-FST, the grant Science and Technology Development Fund, Macao S.A.R FDCT 049/2014/A1 and the grant MYRG2015-00021-FST.

References

- [1] Apostol, T (1974), *Mathematical analysis*, Addison-Wesley.
- [2] Ash, R. B. and Gardner, M. F. (1975). *Topics in Stochastic Processes*, Academic Press, New York.
- [3] Balescu, R. (1997). *Statistical dynamics: matter out of equilibrium*, Imperial College Press, London.
- [4] Bridgman, P. (1940). The Second Law of Thermodynamics and irreversible processes. Phys. Rev. 58, 845
- [5] Chen, Y. and Liu, Y. 2014. On the eigenfunctions of the complex Ornstein-Uhlenbeck operators, Kyoto J. Math., Vol.54, No.3, 577-596.
- [6] Dembo A., Zeitouni O. (2000). Large deviations techniques and applications, Springer-Verlag, New York.
- [7] Deheuvels, P., 2006. Karhunen-Loeve expansions of mean-centered Wiener processes. High Dimensional Probability. IMS Lecture Notes-Monograph Series, vol. 51, pp. 62-76
- [8] Deheuvels, P., 2007. Karhunen-Loeve expansion for a mean-centered Brownian bridge. Statistics Probability Letters, Vol. 77, 1190-1200.

- [9] Deheuvels, P. and Martynov, G. (2003). Karhunen-Loeve expansions for weighted Wiener processes and Brownian bridges via Bessel functions. In: Progress in Probability 55. Birkhauser, Basel., pp. 57-93.
- [10] Eckart, C. (1940). The thermodynamics of irreversible processes. I. the simple fluid. Phys. Rev. 58, 267
- [11] Eckart, C. (1940). The thermodynamics of irreversible processes. II. fluid mixtures. Phys. Rev. 58, 269
- [12] Evans, D.J., Cohen, E.G.D. and Morriss, G.P. (1993). Probability of second law violation in steady flows. Phys. Rev. Lett. **71**, 2401–2404.
- [13] Friz, P.K., Gassiat, P. and Lyons, T. 2015. Physical Brownian motion in a magnetic field as a rough path, Trans. Amer. Math. Soc. 367, 7939-7955.
- [14] Friz, P.K. and Hairer, M. 2014. A Course on Rough Paths, With an Introduction to Regularity Structures, Springer Universitext, Springer.
- [15] Gallavotti, G. and Cohen, E.G.D. (1995). Dynamical ensembles in stationary states. J. Statist. Phys. **80**, 931–970
- [16] Ge, H., Qian, M. and Qian, H. (2012). Stochastic theory of nonequilibrium steady states (Part II): Applications in chemical biophysics. Phys. Rep. 510, 87-118
- [17] Hill, T. L. (1977) Free energy transduction in biology. New York: Academic Press
- [18] Hill, T. L. (1995) Free energy transduction and biochemical cycle kinetics. New York: Springer-Verlag
- [19] Itô K. (1953). Complex Multiple Wiener Integral, Japan J.Math. 22, 63-86
Reprinted in: *Kiyosi Itô selected papers*, Edited by Daniel W. Stroock, S.R.S. Varadhan, Springer-Verlag, 1987.
- [20] V. Jaksic, V. Nersisyan, C.-A. Pillet, A. Shirikyan. Large deviations from a stationary measure for a class of dissipative PDE's with random kicks, Comm. Pure Appl. Math. (2015), to appear.
- [21] Jiang, D. Q., Qian, M. and Qian, M. P. (2004). Mathematical theory of nonequilibrium steady states - On the frontier of probability and dynamical systems. (Lect. Notes Math. 1833) Berlin: Springer-Verlag
- [22] Jiang D.-Q., Qian M., Zhang F.-X. (2003). Entropy production fluctuations of finite Markov chains. Journal of Mathematical Physics, 44(9), 4176.
- [23] Kac M., 1945, Random walk in the presence of absorbing barriers, Ann. Math. Statist., Vol.16, 62-67.

- [24] Kac M. and Siebert A. J. F., 1947, An explicit representation of a stationary Gaussian process, *Ann. Math. Statist.*, Vol.18, 438-442.
- [25] Kifer, Y. Large deviations in dynamical systems and stochastic processes. *Trans. Amer. Math. Soc.* 321 (1990), no. 2, 505-524.
- [26] Kim, W.H. 2011. On the behavior of the EPR of a diffusion process in nonequilibrium steady state. PhD thesis. University of Washington, Seattle.
- [27] Kolmogorov, A.N.: *Zur Theorie der Markoffschen Ketten*, *Math. Ann.*, 112, 155-160 (1936); Translation: K teorii tsepei Markova, *Collection of articles on probability theory and mathematical statistics*, Nauka, Moscow, 1986, 173-178
- [28] Kolmogorov, A.N.: *Zur Umkehrbarkeit der statistischen Naturgesetze*, *Math. Ann.*, 113, 766-772 (1937); Translation: Ob obratimosti statisticheskikh zakonov prirody, *Collection of articles on probability theory and mathematical statistics*, Nauka, Moscow, 1986, 197-204
- [29] Kurchan, J. (1998). Fluctuation theorem for stochastic dynamics. *J. Phys. A: Math. Gen.* **31**, 3719–3729
- [30] Kuo H. H., *Gaussian Measures in Banach Spaces*, LNM No.463, Springer-Verlag, Berlin, 1975.
- [31] Lebowitz, J.L. and Spohn, H. (1999). A Gallavotti-Cohen-type symmetry in the large deviation functional for stochastic dynamics. *J. Statist. Phys.* **95**, 333-365
- [32] Nicolis, G. and Prigogine, I. 1977. *Self-organization in nonequilibrium systems: from dissipative structures to order through fluctuations*. New York: Wiley
- [33] Prigogine, I.: *Introduction to thermodynamics of irreversible processes* (Second ed.). New York: Interscience 1961
- [34] Qian, H. (2001). Mathematical formalism for isothermal linear irreversibility. *Proc. Roy. Soc. A*, 457, 1645-1655
- [35] Qian, M.P., Qian, M. and Gong, G.L. (1991). The reversibility and the entropy production of Markov processes. *Contemp. Math.* 118, 255-261
- [36] Rudin W. 1991. *Functional Analysis*, 2-nd Ed., McGraw-Hill Companies, Inc.
- [37] Seick, E. M., Prabhakar, R., Williams, S. R. and Searles, D. J. (2008). Fluctuation theorems. *Ann. Rev. Phys. Chem.* 59, 603-633
- [38] Simon B., *Trace Ideals and Their Applications*, *Mathematical Surveys and Monographs*, American Mathematical Society; 2 edition (September 2005)
- [39] Wang R., Xu L. 2015, Asymptotics of the Entropy Production Rate for d-Dimensional Ornstein-Uhlenbeck Processes, *Journal of Statistical Physics*.

- [40] L. Wu, Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems, *Stochastic Proc. Appl.* **91** (2001), 205-238.
- [41] Zhang, X. J., Qian, H. and Qian, M. (2012). Stochastic theory of nonequilibrium steady states and its applications (Part I). *Phys. Rep.* 510, 1-86